§9A Cohomology

Version 1.1

We give a short introduction to Čech cohomology and compare it with de Rham cohomology. One major objective is to show that the two cohomology theories are equivalent.

It is helpful but not at all necessary to base everything on sheaves.

We proceed in a twofold way: We first present Čech cohomology with values in an abelian group G. In particular, we compare the cases $G = \mathbb{R}$, $G = \mathbb{C}$ with the de Rham cohomology. We then extend the situation to sheaves. As a result we see that Čech cohomology with values in sheaves is not much more complicated or difficult than the case of Čech cohomology with values in groups, – except for the notion of a sheaf which is a bit involved.

In the following, M will be a topological space which we assume to be paracompact. We are mainly interested in the case of a paracompact manifold.

As a general structure we assume that for all open subsets $U \subset M$ a collection $\mathcal{F}(U)$ of functions or sections of a specified type (for example locally constant, or smooth, or continuous, or holomorphic, etc.) is given with certain compatibility conditions with respect to the inclusion $U \subset V$, $U, V \subset M$ open. The general case leads to sheaves. The quite restricted but already interesting case is the following:

Let G be a fixed abelian group and for open $U \subset M$

$$\mathcal{F}(U) = \mathcal{F}(U, G) := \{ g : U \to G \mid g \text{ locally constant} \}$$

Of course $\mathcal{F}(U) = \mathcal{C}(U, G) = \{ g : U \to G \mid \text{continuous} \}$ if G is endowed with the discrete topology (where all subsets $H \subset G$ are open).

$\mathcal{F}(U)$ is an abelian group for each open subset $U \subset M$ by pointwise multiplication or addition depending of whether the composition in G is written multiplicatively or additively. We choose the additive notation in the following. To every inclusion

$$V \subset U \quad \text{of open subsets } U, V \subset M$$

there corresponds the natural RESTRICTION map

$$\rho_{V,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \, , \quad g \longmapsto g|_V$$

which is a homomorphism of groups with the property

(9A.1) LEMMA:

$$\varrho_{W,V} \circ \varrho_{V,U} = \varrho_{W,U} \quad \& \quad \varrho_{U,U} = id_{\mathcal{F}(U)}$$

for open subsets $W, V, U \subset M$ with $W \subset V \subset U$.

Remarks: Already the case $G = \mathbb{R}$ is interesting. It leads to the deRham cohomology. When the abelian group E is a vector space over $\mathbb{R}$, the groups $\mathcal{F}(U) = \mathcal{F}(U, E)$ are $\mathbb{R}$-vector spaces as well and the restrictions $\varrho_{U,V}$ are linear maps.

(9A.2) DEFINITION: Now let $\mathcal{U} = (U_j)_{j \in I}$ be an open cover of M. A q-SIMPLEX σ ($q \in \mathbb{N}$) is an ordered $(q+1)$ tuple $(U_{j_0}, U_{j_1}, \dots, U_{j_q})$, $j_i \in I$, such that $U_{i_0 \dots i_q} := U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_q} \neq \emptyset$. $|\sigma| = U_{i_0 \dots i_q}$ is the SUPPORT of σ. $\Sigma(q)$ denotes the set of q simplices.
The k-th PARTIAL BOUNDARY of σ, $0 \leq k \leq q$, is the $(q-1)$-simplex

$$\partial_j \sigma := (U_{i_0}, \dots, \hat{U}_{i_k}, \dots U_{i_q})$$

and the BOUNDARY $\partial\sigma$ of σ is

$$\partial\sigma := \sum_{k=0}^{q} (-1)^{k+1} \partial_k \sigma .$$

(We allow finite formal sums of simplices.)

This definition is based only on the topology of M and will be needed also for the sheaf cohomology.

In the next definition we have to fix an abelian group G and we consider the induced groups $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{U}, G)$ and the restriction homomorphisms

$$\varrho_{V,\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V), \quad V \subset \mathcal{U} \subset M.$$

(9A.3) DEFINITION:

1° A q-COCHAIN of $\mathfrak{U}$ with coefficients in $\mathcal{F}$ (or in G) is a family $\eta = (\eta_\sigma)_{\sigma \in \Sigma(q)}$, $\eta_\sigma \in \mathcal{F}(|\sigma|)$, or a map

$$\sigma \mapsto \eta(\sigma) \in \mathcal{F}(|\sigma|), \quad \sigma \text{ a } q\text{-simplex of } \mathfrak{U},$$

$C^q(\mathfrak{U}, \mathcal{F}) = C^q(\mathfrak{U}, G)$ is the abelian group of q-cochains.

2° The COBOUNDARY OPERATOR $\delta = \delta_q$

$$\delta : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})$$

is

$$\eta \mapsto \delta\eta, \quad \delta\eta(\sigma) := \sum_{k=0}^{q+1} (-1)^k \operatorname{restr}|_{|\sigma|} \eta(\partial_j \sigma).$$

A q-cochain η with $\delta\eta = 0$ is called a q-COCYCLE.

It is easy to show that δ is a homomorphism and that $\delta^2 = 0$.[*] Hence, the following notations make sense

$$Z^q(\mathfrak{U}, G) := \operatorname{Ker}\left(\delta_q : C^q(\mathfrak{U}, G) \to C^{q+1}(\mathfrak{U}, G)\right),$$

$$B^q(\mathfrak{U}, G) := \operatorname{Im}\left(\delta_{q-1} : C^{q-1}(\mathfrak{U}, G) \to C^{q+1}(\mathfrak{U}, G)\right),$$

$$\check{H}^q(\mathfrak{U}, G) := Z^q(\mathfrak{U}, G) / B^q(\mathfrak{U}, G).$$

[*] Übung!

Let $\mathfrak{W} = (V_k)_{k \in K}$ be another open cover of M such that for each $k \in K$ there exists a $j(k) \in I$ with

$$V_k \subset U_{j(k)} .$$

Then $\mathfrak{W}$ is called a REFINEMENT of $\mathfrak{U}$ and $j : K \to I$ is a REFINEMENT MAP, yielding a partial order $\mathfrak{U} \ll \mathfrak{W}$. This refinement map induces a homomorphism

$$\tilde{j} : C^p(\mathfrak{U}, G) \longrightarrow C^p(\mathfrak{W}, G)$$

for each $p \in \mathbb{N}$ by restricting: To $\sigma = (V_{k_0}, \dots, V_{k_p})$ there corresponds $\sigma' = (U_{j(k_0)}, \dots, U_{j(k_p)})$ and

$$\tilde{j}(\eta)(\sigma) = \mathrm{restr}|_{|\sigma|}\, \eta(\sigma') .$$

Because of $\tilde{j}(Z^p(\mathfrak{U}, G)) \subset Z^p(\mathfrak{W}, G)$ and $\tilde{j}(B^p(\mathfrak{U}, G)) \subset B^p(\mathfrak{W}, G)$ the restriction map $\tilde{j}$ descends to a homomorphism

$$\tilde{j} : H^p(\mathfrak{U}, G) \longrightarrow H^p(\mathfrak{W}, G) .$$

In this way we obtain an inductive (or direct) system of abelian groups $\left(H^q(\mathfrak{U}, G)\right)_{\mathfrak{U} \ll \mathfrak{W}}$. The direct limit of this direct system is – by definition – the p^{th} ČECH COHOMOLOGY GROUP

$$\check{H}^p(M, G) := \lim_{\mathfrak{U} \ll \mathfrak{W}} \mathrm{dir}\; H^p(\mathfrak{U}, G) = \varinjlim H^p(\mathfrak{U}, G)$$

(9A.4) DEFINITION: $\check{H}^q(M,G)$ is the q^{th} ČECH COHOMOLOGY GROUP on M with values in the abelian group G.

Fortunately, one can avoid to consider the limit in concrete calculations: There exist enough open covers $\mathfrak{U}$ of M such that

$$H^p(X,G) = H^p(\mathfrak{U},G) \quad \text{(independently of } G\text{)}.$$

(More precisely, the canonical $H^p(\mathfrak{U},\mathcal{F}) \to H^p(M,\mathcal{F})$ is an isomorphism.) Such covers are called LERAY COVERS. In the case of a manifold M any open cover $\mathfrak{U} = (U_j)_{j\in I}$ where all intersections $U_i \cap U_j \cap \ldots U_k$ are diffeomorphic to convex open subsets of $\mathbb{R}^n$ is a Leray cover.[*] The same holds if $U_i \cap \ldots \cap U_k$ are contractible.

Let us now come to the comparison of $\check{H}^p(\mathfrak{U},\mathbb{R})$ with the de Rham cohomology groups $H^p_{dR}(M,\mathbb{R})$:

$$H^p_{dR}(M,\mathbb{C}) = \{\alpha \in \Omega^p(M) \mid d\alpha = 0\} \big/ \{\alpha \in \Omega^p(M) \mid \exists \beta : d\beta = \alpha\}$$

(9A.5) PROPOSITION: Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of a smooth manifold M such that all intersections $U_{j_0 j_1 \ldots j_p} = U_{j_0} \cap U_{j_1} \cap \ldots \cap U_{j_p}$ are empty or diffeomorphic to a convex open subset $\subset \mathbb{R}^n$. Then there exists a natural isomorphism

$$H^q_{dR}(M,\mathbb{R}) \longrightarrow H^q(\mathfrak{U},\mathbb{R})$$

[*] Übung!

for each $q \in \mathbb{N}$.

□ <u>Proof</u>: Note that there always exist such open covers.[*] Here are the details of proof for the case $p = 1$: Let $\alpha \in \Omega^1(M)$ with $d\alpha = 0$. Then $\alpha|_{U_j}$ is exact by the Lemma of Poincaré (since U_j is diffeomorphic to a convex open subset). Let $f_j \in \mathcal{E}(U_j)$ with $df_j = \alpha|_{U_j}$. Then $\eta_{ij} := (f_i - f_j)|_{U_{ij}}$ is a constant $\eta_{ij} \in \mathbb{R}$ since

$$d(f_i - f_j)|_{U_{ij}} = \alpha|_{U_{ij}} - \alpha|_{U_{ij}} = 0.$$

The 1-cochain $\eta := (\eta_{ij})$ satisfies

$$(\delta\eta)_{ijk} = \eta_{ij} + \eta_{jk} + \eta_{ki} = f_i - f_j + f_j - f_k + f_k - f_i = 0,$$

hence η is a cocycle $\eta \in C^1(\mathfrak{U}, \mathbb{R})$ and defines an element

$$[\eta] \in H^1(\mathfrak{U}, \mathbb{R}) = H^1(M, \mathbb{R}).$$

To what extent is this cohomology element $[\eta]$ independent of the various choices made?

Let $\alpha^* \in \Omega^1(M)$ be in the same deRham class as α, i.e. $\alpha^* - \alpha = dg$, $g \in \mathcal{E}(M)$. Choose $f_j^* \in \mathcal{E}(U_j)$ with $df_j^* = \alpha^*|_{U_j}$ and $\eta^*_{ij} = (f_i^* - f_j^*)|_{U_{ij}}$. By $d(f_j^* - f_j) = (\alpha^* - \alpha)|_{U_j} = dg|_{U_j}$ we can write

$$f_j^* = f_j + g + c_j$$

with suitable constants $c_j \in \mathbb{R}$. Hence,

[*] Übung. "$U_{i_0 \ldots i_q} = \emptyset$ or contractible" is easier to construct!

$$\eta^*_{ij} = \eta_{ij} + c_i - c_j \quad , \text{ i.e. } \delta c = \eta^* - \eta ,$$

when $c = (c_i)$. As a result, $[\eta^*] = [\eta] \in H^1(\mathfrak{U}, \mathbb{R})$ and the map

$$H^1_{dR}(M, \mathbb{R}) \ni [\alpha] \longmapsto [\eta] = \Psi(\alpha) \in H^1(M, \mathbb{R})$$

is well-defined. Evidently Ψ is a homomorphism which is injective. To show surjectivity let $\eta \in H^1(\mathfrak{U}, \mathbb{R})$. We find a smooth partition of unity $(h_k)_{k \in I}$ with supp h_j compact and supp $h_j \subset U_j$ which is locally finite. Define

$$\alpha_\eta := \sum_{i,j} \eta_{ij} h_i dh_j \quad , \quad \eta = (\eta_{ij})_{ij} \in Z^1(\mathfrak{U}, \mathbb{R}).$$

Then
$$d\alpha_\eta = d\Big(\sum_{i,j} \eta_{ij} h_i\Big) dh_j = \sum_{i,j} \eta_{ij}\, dh_i \wedge dh_j$$
$$= \sum_{i,j,k} (\delta\eta)_{ijk}\, h_k\, dh_i \wedge dh_j \quad , \text{ since } \sum h_k = 1 .$$

(All sums are finite since $(h_k)_{k \in I}$ is locally finite.) Hence $d\alpha_\eta = 0$, $\alpha_\eta|_{U_k} = df_k$, where

$$f_k = \sum \eta_{kj} h_j \text{ (see below)},$$

hence $f_k - f_\ell = \sum (\eta_{kj} - \eta_{\ell j}) h_j$, $\eta_{kj} - \eta_{\ell j} = \eta_{k\ell}$ $(\delta\eta = 0)$. We conclude

$$f_k - f_\ell = \sum_j \eta_{k\ell} h_j = \eta_{k\ell} \sum_j h_j = \eta_{k\ell} ,$$

i.e. $\Psi(\alpha_\eta) = [\eta] \in H^1(\mathfrak{U}, \mathbb{R})$.

In order to show $d\left(\sum_j \eta_{kj} h_j\right) = \alpha_\eta|_{U_k}$:

$$
\begin{aligned}
\alpha_\eta &= \sum_j \sum_{i \neq k} \eta_{ij} h_i \, dh_j + \sum_j \eta_{kj} h_k \, dh_j \\
&= \sum_j \sum_{i \neq k} \eta_{ij} h_i \, dh_j + \sum_j \eta_{kj} \left(1 - \sum_{i \neq k} h_i\right) dh_j \\
&= \sum_j \sum_{i \neq k} \left(\eta_{ij} + \eta_{jk}\right) h_i \, dh_j + \sum_j \eta_{kj} \, dh_j \\
&= \sum_j \left(\sum_{i \neq k} \eta_{ik} h_i\right) dh_j + d\left(\sum \eta_{kj} h_j\right) \\
&= \left(\sum_{i \neq k} \eta_{ik} h_i\right) \sum_j dh_j + d\left(\sum \eta_{kj} h_j\right) \quad \& \quad d\left(\sum h_j\right) = 1 \\
&= d\left(\sum \eta_{kj} h_j\right).
\end{aligned}
$$

We have shown that $\Psi : H^1(M, \mathbb{R}) \to H^1(\mathfrak{U}, \mathbb{R})$ is an isomorphism.

The proof extends directly to cases $p \neq 1$. The main part is again the surjectivity with help of a smooth partition of unity (h_k). To $\eta \in Z^p(\mathfrak{U}, \mathbb{R})$ we define

$$
\alpha_\eta := \sum \eta_{i_0 i_1 \dots i_p} h_{i_0} \, dh_{i_1} \wedge \dots \wedge dh_{i_p}
$$

and see $\Psi(\alpha_\eta) = [\eta]$. The definition of Ψ can be done as before by descending from $\alpha \in \Omega^p(M)$, $d\alpha = 0$, to $\beta_j \in \Omega^{p-1}(M)$ with $d\beta_j = \alpha|_{U_j}$, $\gamma_{ij} \in \Omega^{p-2}(M)$, with $d\gamma_{ij} = \beta_i - \beta_j \,|_{U_{ij}}$ etc. $\square$

We want to explain this definition of Ψ in the case of $p = 2$ in order to comment the integrality condition in the form we need it in the next section.

(9A.6) REMARK: (Integrality condition). The natural isomorphism

$$\Psi : H^2_{dR}(M,\mathbb{R}) \to \check{H}^2(M,\mathbb{R})$$

can be constructed as follows. Choose an open cover $\mathfrak{U} = (U_j)$ as before in the last proposition. To a closed $\alpha \in \Omega^2(M)$ we find $\beta_j \in \Omega^1(M)$ with $d\beta_j = \alpha|U_j$ and functions $f_{ij} \in \mathcal{E}(U_{ij})$ with $df_{ij} = \beta_i - \beta_j | U_{ij}$. Hence,

$$\eta_{ijk} := f_{ij} + f_{jk} + f_{ki} \in \mathbb{R}$$

is constant and defines $\eta = (\eta_{ijk}) \in Z^2(\mathfrak{U},\mathbb{R})$. The class $[\eta] = \Psi(\alpha)$ is independent of the choices α, β_j, f_{ij} and yields the isomorphism

$$\Psi : H^2_{dR}(M,\mathbb{R}) \to H^2(\mathfrak{U},\mathbb{R}) = \check{H}^2(M,\mathbb{R}).$$

The integrality condition in section 6 (cf. [G1]–[G3]) can now be reformulated in a complete and proper way.

The G3 – Version (cf. sect. 6) means that the choices can be made in such a way that $\eta_{ijk} \in \mathbb{Z}$.
More explicitly:

(9A.7) DEFINITION: A closed $\omega \in \Omega^2(M)$ is ENTIRE (or ω resp. its deRham class $[\omega] \in H^2_{dR}(M,\mathbb{R})$ satisfies the INTEGRALITY CONDITION) if

[G3] There exists an open cover $(U_j)_{j\in I}$ of M such that the class $[\omega] \in H^2_{dR}(M,\mathbb{C})$ contains (as a Čech class $[\omega] \in H^2((U_j)_{j\in I},\mathbb{C}) \cong \check{H}^2(M,\mathbb{C})$) a cocycle $c = (c_{ijk})$, with $c_{ijk} \in \mathbb{Z}$ for all $i,j,k \in I$ with $U_{ijk} \neq \emptyset$.

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We now turn our attention to sheaf cohomology (à la Čech).

(9A.8) <u>DEFINITION</u> : For a topological space M we always have the category $t(M)$ of open subsets. The objects are the open subsets and the morphisms are the inclusions $U \subset V$, $U, V \in M$ open.

A PRESHEAF of abelian groups on M is a contravariant functor

$$\mathcal{F} : t(M) \longrightarrow Ab$$

from $t(M)$ into the category of abelian groups Ab.

Hence, $\mathcal{F}(U)$ is an abelian group for each $U \subset M$ open and to every inclusion $V \subset U$ there corresponds a homomorphism

$$\rho_{V,U} = \mathcal{F}(V,U) : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that

$$\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U} \quad \& \quad \rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$$

for open $W \subset V \subset U$.

(9A.9) EXAMPLES:

1° G an abelian group, $\mathcal{F}(U) := \{f : U \to G \mid f \text{ a map}\}$ and $\rho_{V,U}(f) = \mathrm{restr}|_V f = f|_V$.

2° G as before, $\mathcal{F}(U) := \{f : U \to G \mid f \text{ locally constant}\}$. This is the case we have studied in the first part of this section. See, in particular, Lemma (9A.1).

3° $\mathcal{F}(U) = \mathcal{C}(U) = \mathcal{C}(U, \mathbb{R})$ and ρ restriction as as before.

4° Let G be a topological group and abelian. Then $\mathcal{F}(U) = \mathcal{C}(U, G)$ and ρ restriction as before defines a pre-sheaf, generalizing 3°.

5° $\mathcal{F}(U) = \mathcal{C}^\infty(U) = \mathcal{E}(U)$.

6° $\mathcal{F}(U) = \mathcal{O}(U)$ if M is a complex manifold and $\mathcal{O}(U)$ is the space of holomorphic fctns on U.

7° $\mathcal{F}(U) = \Gamma(U, L)$ for a line bundle over M.

8° $\mathcal{F}(U) = \Omega^p(U)$, $U \subset M$ open subset of a smooth manifold. etc.

In most of the examples $\mathcal{F}(U)$ is a vector space, in some cases an algebra, or a module over another presheaf, like Ω^p is a presheaf of $\mathcal{E}$-modules.

(9A.10) <u>DEFINITION:</u> A presheaf $\mathcal{F}$ is a SHEAF if for all open subsets $U \subset M$ and all open covers $\mathcal{U} = (U_j)_{j \in I}$ of U the following property is satisfied:

Any collection $f_i \in \mathcal{F}(U_i)$, $i \in I$, is of the form

$$f_i = \rho_{U_i, U}(f) \text{ for all } i \in I,$$

for a unique element $f \in \mathcal{F}(U)$ if and only if for all $i, j \in I$ the compatibility property

$$\rho_{U_i \cap U_j, U_i}(f_i) = \rho_{U_j \cap U_i, U_j}(f_j)$$

holds.

We know this property from continuous (or smooth) functions. If a collection of maps

$$f_i : U_i \to G$$

is continuous and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then the map

$$f(a) := f_i(a), \quad a \in U_i,$$

is a well-defined continuous map with

$$f|_{U_i} = f_i, \quad i \in I.$$

Moreover, f is unique.

All the examples in (9A.9) are sheaves. The "constant" presheaf $\mathcal{F}(U) = G$, $U \subset M$ open, and $\rho_{V,U} = id_G : G \to G$ is in general not a sheaf. Why? [*]

(9A.11) DEFINITION: A q-COCHAIN of $\mathfrak{U}$ with coefficients in $\mathcal{F}$ is a map

$$\sigma \mapsto \eta(\sigma) \in \mathcal{F}(|\sigma|), \quad \sigma \text{ a } q\text{-simplex of } \mathfrak{U},$$

$C^q(\mathfrak{U}, \mathcal{F})$ is the abelian group of q-cochains.

The COBOUNDARY OPERATOR

$$\delta : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F}), \quad \delta = \delta_q,$$

is

$$\eta \mapsto \delta\eta, \quad \delta\eta(\sigma) = \sum_{k=0}^{q+1} (-1)^k \operatorname{restr}|_{|\sigma|} \eta(\partial_j \sigma).$$

It is easy to show $\delta^2 = 0$. δ is a homomorphism.

$$Z^q(\mathfrak{U}, \mathcal{F}) = \operatorname{Ker}\left(\delta_q : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})\right)$$

$$B^q(\mathfrak{U}, \mathcal{F}) = \operatorname{Im}\left(\delta_{q-1} : C^{q-1}(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})\right)$$

$$\check{H}^q(\mathfrak{U}, \mathcal{F}) = Z^q(\mathfrak{U}, \mathcal{F}) / B^q(\mathfrak{U}, \mathcal{F})$$

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[*] Übung: Give conditions on M and/or G.

Finally, if $\mathfrak{W} = (V_k)_{k \in K}$ is a refinement of $\mathfrak{U}$ with refinement map $j : K \to I$, then we have again an induced homomorphism

$$\tilde{j} : H^q(\mathfrak{U}, \mathcal{F}) \longrightarrow H^q(\mathfrak{W}, \mathcal{F}) .$$

The corresponding direct limit gives the cohomology:

(9A.12) <u>DEFINITION:</u> The q^{th} ČECH COHOMOLOGY GROUP on M with values in the sheaf $\mathcal{F}$ on M is defined by

$$\check{H}^q(M, \mathcal{F}) = \varinjlim H^q(\mathfrak{W}, \mathcal{F}) .$$

Again there are Leray covers which reduce the calculation of $\check{H}^q(M, \mathcal{F})$ to that of $H^q(\mathfrak{U}, \mathcal{F})$.